Formation of singularities on the free surface of an ideal fluid

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It is shown that the equations of motion of an ideal fluid with a free surface in the absence of both gravitational and capillary forces can be effectively solved in the approximation of small surface angles. It can be done by means of an analytical continuation of both the velocity potential on the surface and its elevation. For almost arbitrary initial conditions the system evolves to the formation of singularities in a finite time. Three kinds of singularities are shown to be possible. The first one is of the root character provided by the analytical behavior of the velocity potential. In this case the process of the singularity formation, representing some analog of the wave breaking, is described as a motion of branch points in the complex plane towards the real axis. The second type can be obtained as a result of the interaction of two movable branch points leading to the formation of wedges on the free surface. The third kind is associated with a motion in the complex plane of the singular points of the analytical continuation of the elevation, resulting in the appearance of strong singularities for the surface profile.

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I. INTRODUCTION

The formation of the singularities in the wave system in a finite time, or by other words—the wave collapse, is one of the basic phenomena in nonlinear physics. The collapses play an essential role in various fields of physics. In many cases the collapse is the most effective mechanism of the wave-energy dissipation.

From the mathematical point of view, collapse means that the solution of the Cauchy problem for some evolution PDE (partial differential equation) exists only for finite time until some definite moment $t = t_0$ and cannot be continued for $t > t_0$. At the moment $t = t_0$ the solution loses its initial smoothness and a singularity appears. What kind of singularities will arise depends on the physical model. For example, for the self-focusing of light [1] or for the collapse of Langmuir waves [2] the amplitude of electromagnetic waves tends to the infinity. In other words, that is the wave breaking in gas dynamics described by the well-known Riemann solution (see, for example, [3]), the first derivative of the velocity becomes infinite at the breaking moment of time. For sea surface waves the analogous phenomenon leads to the infinite second derivative of the surface profile (so that angles or cones appear on the surface). Checking analyticity violation is the most sensitive tool for studying that set of collapses. Loss of analyticity of vortex sheets at the nonlinear stage of the Kelvin-Helmholz instability [4] is such an example.

In this paper we shall consider how the singularities appear as a result of the analyticity breaking on the free surface of an ideal liquid in the absence of both gravity and surface tension. The problem seems to be somehow artificially formulated. This question is very important, nevertheless, for understanding the evolution of the boundary between two fluids while studying sea surface waves and the nonlinear stage of the Rayleigh-Taylor instability resulting in the finger structure (see, for instance, [5], and references therein). The problem in this statement was formulated [6] by one of the authors (V.Z.) of the present paper. It was assumed that the singularity formation on the free surface of the ideal fluid or in a more general case, for the boundary between two ideal fluids, is mainly connected with inertial forces; other factors give minor correction. This means that if one considers, for instance, motion of the ideal liquid drop (without both gravity and surface tension) then on the surface of the drop there will appear a singularity of the wedge type. This idea was later confirmed by direct numerical integration of the Euler equation for the case of deep water [7]. In this paper we present the analytical solution of this problem based both on the perturbation approach, assuming small angles of the surface variations, and on using the Hamiltonian formalism for the description of the surface motion. The main conjecture of this paper is as follows. The formation of singularities on the free surface for small-angle approximation can be considered as the process of the wave breaking in the complex plane where the solution can be extended to. This results in the motion of both branch points of the analytical continuation of the velocity potential and of singular points of the analytical extension of the surface elevation. When for the first time the most "rapid" singular point will reach the real axis it will be just the singularity appearance. Respectively, three possible kinds of singularities are shown to arise. For those of the first kind, the first derivative of the tangent velocity on the surface turns into infinity at the touching moment of time. So does the second space derivative of the free surface coordinate $z = \eta(x,t)$, i.e., η_{xx}, will also become infinite. These are the weak singularities of the root character ($\eta_{xx} \sim |x|^{-1/2}$) which can be assumed to serve as the origin of the more powerful singularities observed in the numerical experiments [7] or to represent the separate type of the singularities. These kinds of singularities prove to be consistent with an assumption about small surface angles. It is shown that the interaction of two movable branch points of the tangent velocity can lead under some definite conditions to the formation of the second type of singularities—wedges on the surface shape. Close to the collapse time the self-similar solution for such singularities occurs to be compatible with the complete system of equations describing arbitrary angle values. The third type is caused by the initial analytical properties of $\eta_0(x)$ resulting in the formation of strong singular surface profile.

II. BASIC EQUATIONS

Let us consider an ideal fluid with a free surface, $z = \eta(x,y,t)$, that occupies the region $-\infty < z \leq \eta(x,y,t)$. We will be interested in an irrotational motion of a liquid, implying the liquid velocity to be a potential one, $\mathbf{v} = \nabla\Phi$. In the absence of external fields (e.g., gravitational) the potential Φ satisfies the nonstationary Bernoulli equation,

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\Phi)^2 + p = 0, \tag{1}$$

which combines in a complete closed system, when amplified with the incompressibility equation

$$\Delta\Phi = 0, \tag{2}$$

the kinematic relation on the free surface

$$\frac{\partial\eta}{\partial t} = \left(\frac{\partial\Phi}{\partial z} - \nabla\eta\nabla\Phi\right)\Big|_{z=\eta} = v_n\sqrt{1+(\nabla\eta)^2}, \tag{3}$$

and boundary conditions:

$$p\,|_{z=\eta} = 0\ , \tag{4}$$

$$\Phi\,|_{z\to-\infty} \to 0. \tag{5}$$

Here v_n is the velocity component, normal to the free surface $z = \eta(x,y,t)$. We have also assumed here that the capillarity is absent.

Under these assumptions, let us, following Zakharov [8], rewrite Eqs. (1)—(5) in the Hamiltonian form. For this purpose, one should introduce the function

$$\Psi(x,y,t) = \Phi(x,y,\eta(x,y,t),t), \tag{6}$$

which presents the value of the potential Φ on the free surface $z = \eta(x,y,t)$. The quantities $\Psi(x,y,t)$ and $\eta(x,y,t)$ are canonically conjugated, so that the equations of motion take the standard Hamiltonian form [8];

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \tag{7}$$

$$\frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \tag{8}$$

where the Hamiltonian

$$H = \frac{1}{2}\int d\mathbf{r}_\perp \int_{-\infty}^{\eta} dz\ (\nabla\Phi)^2 \tag{9}$$

coincides with the total (kinetic) energy of a liquid. The potential Φ (and consequently $\nabla\Phi$), being the solution of the Laplace equation (2) with boundary conditions (5) and (6), represents some functional of Ψ and η and can be determined with the help of the corresponding Green function.

Let us assume $|\nabla\eta| \ll 1$; that means surface perturbations are fairly flat. In such a case one can use a usual perturbation theory to expand the quantities Φ and $\nabla\Phi$ in series with respect to $|\nabla\eta|$. Respectively, the Hamiltonian itself can be represented as an expansion in a power series of canonical variables. We will restrict ourselves only by quadratic and cubic terms in the Hamiltonian. In order to find them it is convenient to rewrite H as the integral over the free surface:

$$\begin{aligned} H &= \frac{1}{2}\int d\mathbf{r}_\perp\ \Psi v_n\left(1+(\nabla\eta)^2\right)^{1/2} \\ &= \frac{1}{2}\int d\mathbf{r}_\perp\ \Psi\left(\frac{\partial\Phi}{\partial z} - \nabla\eta\nabla_\perp\Phi\right). \end{aligned}$$

In this approximation it is sufficient to substitute $\nabla_\perp\Phi$ in the Hamiltonian by $\nabla\Psi$. Thus, the only term $\frac{\partial\Phi}{\partial z}$ remains to be expressed through Ψ and η. To find it let us make the Fourier transform in (2) with respect to $\mathbf{r}_\perp = (x,y)$. As a result, the Fourier transform of the solution Φ to the Laplace equation (2), subject to condition (5) at the infinity, is readily found as the following: $\Phi_{\mathbf{k}}(z) = A_{\mathbf{k}}e^{kz}$. Consequently,

$$\frac{\partial\Phi_{\mathbf{k}}(z)}{\partial z} = kA_{\mathbf{k}}e^{kz}. \tag{10}$$

Using definition (6) we find with the needed accuracy

$$\Psi_{\mathbf{k}} = A_{\mathbf{k}} + \int k_1A_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d\mathbf{k}_1\ d\mathbf{k}_2,$$

that yields after one iteration

$$A_{\mathbf{k}} = \Psi_{\mathbf{k}} - \int k_1\Psi_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d\mathbf{k}_1\ d\mathbf{k}_2. \tag{11}$$

The same procedure, making use of relations (10) and (11), enables to get the Fourier transform of the derivative $\frac{\partial\Phi}{\partial z}|_{z=\eta} = \chi$:

$$\chi_{\mathbf{k}} = k\Psi_{\mathbf{k}} + \int (k_1^2 - kk_1)\Psi_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d\mathbf{k}_1\, d\mathbf{k}_2. \tag{12}$$

Substitution of (12) into the Hamiltonian H (9) and consequent integration by parts give

$$H = \frac{1}{2}\int \Psi\hat{k}\Psi d\mathbf{r} + \frac{1}{2}\int \left[(\nabla\Psi)^2 - (\hat{k}\Psi)^2\right]\eta\, d\mathbf{r}. \tag{13}$$

Here $\hat{k}$ is the integral operator with the difference kernel, whose Fourier transform is modulus of the wave vector $\mathbf{k}$.

The equations of motion (7) and (8), corresponding to Hamiltonian (13), acquire the following form:

$$\frac{\partial\eta}{\partial t} = \hat{k}\Psi - \left[\hat{k}(\eta\hat{k}\Psi) + \nabla(\eta\nabla\Psi)\right], \tag{14}$$

$$\frac{\partial\Psi}{\partial t} = \frac{1}{2}\left[(\hat{k}\Psi)^2 - (\nabla\Psi)^2\right]. \tag{15}$$

The remarkable property of these equations is the splitting off Eq. (15), which involves only variable Ψ, from that of (14), which governs the behavior of elevation η. Such a separation is a peculiarity of the used perturbation order and is being lost in next orders, when η appears in Eq. (15) as well. Note also that since we assume $|\nabla\eta| \ll 1$, it is possible to omit the second term in the rhs of Eq. (14):

$$\frac{\partial\eta}{\partial t} = \hat{k}\Psi. \tag{16}$$

For the sake of simplicity consider the one-dimensional case when functions Ψ and η depend only on x (and t) and the integral operator $\hat{k}$ may be presented in the form

$$\hat{k} = -\frac{\partial}{\partial x}\hat{H},$$

where

$$(\hat{H}f)(x) = \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{+\infty}\frac{f(x')}{x'-x}\,dx'$$

is the Hilbert transform (P.V. denotes principle value of the integral). Then, it is convenient to introduce a new function $v = \partial\Psi/\partial x$, which has a meaning of the tangent velocity on the free surface. As a result, Eqs. (15) and (16) can be rewritten as

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left[(\hat{H}v)^2 - v^2\right], \tag{17}$$

$$\frac{\partial\eta}{\partial t} = -\,\hat{H}v. \tag{18}$$

Remember that the Hilbert transform acts on a function, analytic in the upper (lower) complex half-plane, as multiplied by $(+i)$ [respectively, by $(-i)$]. Such a property implies that v can be represented as a sum of two functions

$$v = v^{(+)} + v^{(-)}, \tag{19}$$

where $v^{(\pm)} = \hat{H}^{(\pm)}v$ is a function, analytically extendable into the upper (lower) half-plane of complex variable x, respectively, and $\hat{P}^{(\pm)} = \frac{1}{2}(1 \mp i\hat{H})$ are projectors. Then

$$\hat{H}v = i(v^{(+)} - v^{(-)}), \tag{20}$$

provided $v \to 0$ at $|x| \to \infty$. After substitution of relations (19) and (20) into Eq. (17), the latter decomposes into separate equations for $v^{(+)}$ and $v^{(-)}$:

$$\frac{\partial v^{(\pm)}}{\partial t} + 2v^{(\pm)}\frac{\partial v^{(\pm)}}{\partial x} = 0. \tag{21}$$

Equations (21) can be solved by the standard method of characteristics:

$$v^{(\pm)} = F^{(\pm)}(x_0), \tag{22}$$

$$x = x_0 + 2F^{(\pm)}(x_0)t, \tag{23}$$

where functions $F^{(\pm)}$ are defined from initial conditions. According to (19) and (20), functions $v^{(\pm)}$ are complex conjugates on the real axis ($\text{Im}x = 0$), so it is enough to find a solution only for $v^{(+)}$. A shape of surface η has to be found from the equation

$$\frac{\partial\eta}{\partial t} = -i(v^{(+)} - v^{(-)}). \tag{24}$$

III. DYNAMICS OF SIMPLE SINGULARITIES

Let us show on a simple example that Eqs. (21) and (24) describe formation of a singularity in a finite time. Let $F^{(+)}(x_0)$ be a rational function with one simple pole in the lower half-plane,

$$F^{(+)}(x_0) = \frac{A}{x_0 + ia},$$

where $\text{Re}a > 0$. Then the dependence $x_0 = x_0(x,t)$ can be readily found by means of (23),

$$x_0 = \frac{1}{2}(x - ia) + \sqrt{\frac{1}{4}(x + ia)^2 - 2At} \tag{25}$$

that yields the solution of (21) in the form (22)

$$v^{(+)}(x,t) = \frac{2A}{x + ia + \sqrt{(x+ia)^2 - 8At}}. \tag{26}$$

Thus, instead of the initial pole at the point $x = -ia$ there appears a cut, connecting two moving branch points

$$x_{1,2} = -ia \pm 2\sqrt{2At}. \tag{27}$$

It should be noted that the branch of the square root in (26) [or in (25)] has to be defined in such a way that function (26) would have asymptotics $v^{(+)}(x,t) \to A/x$ as $|x| \to \infty$.

It is seen from (27) that if $A > 0$, the cut will expend over time t parallel to the real axis. In any other case the points $x_{1,2}(t)$ (27) will move under some angle to the real axis. For this case there exists such a moment t_0 when the branch point reaches the real axis, and thereafter the solution of the problem breaks down. For example, if $A = -1/8$, then the cut will spread along the imaginary axis and reach the real axis at the breaking moment of time $t = t_0 = 1$ (for $a = 1$). We will consider just this partial solution because as it will be seen further this situation does not differ in essence from a general one.

Now let us find what kinds of singularities occur close to the moment of the cut touching the real axis. First of all show that the profile of $v = v^{(+)} + v^{(-)}$ breaks after $t = t_0$ and becomes ambiguous. It means that for $t \to t_0$ $\frac{\partial v}{\partial x} \to \infty$ at some point $x = x_{\rm br}$. For $A = -1/8$ and $a = 1$, $x_{\rm br} = 0$.

As it folows from (26) in such a case

$$\frac{\partial v}{\partial x} = \frac{1}{2}\mathrm{Re}\left\{(x+i)^2 + t + (x+i)\sqrt{(x+i)^2+t}\right\}^{-1}. \tag{28}$$

It is evident that at the point $x = 0$ this derivative behaves as

$$\left.\frac{\partial v}{\partial x}\right|_{x=0} \approx -\frac{1}{\sqrt{\tau}}, \tag{29}$$

where $\tau = t_0 - t = 1 - t$. At the vicinity of $\tau = 0$ and $x = 0$ expression (28) can be represented in the form

$$\frac{\partial v}{\partial x} \approx -\frac{1}{\sqrt{4x^2+\tau^2}}\left[\frac{1}{2}\left(\tau + \sqrt{4x^2+\tau^2}\right)\right]^{1/2}. \tag{30}$$

Thus, at the critical moment of $\tau = 0$ the velocity derivative looks like

$$\frac{\partial v}{\partial x} \approx -\frac{1}{2}|x|^{-1/2}. \tag{31}$$

For the imaginary part of $v^{(+)}$ the formula, corresponding to (29), has the following form:

$$n = 2\mathrm{Im}v^{(+)}|_{x=0} = \frac{1}{1+\sqrt{\tau}}; \tag{32}$$

this means that this value grows and reaches its maximum at $\tau = 0$. Note that expression (32) corresponds to the maximum of n as a function of x.

In order to determine elevation η, substitute $v^{(+)}$ (26) into Eq. (18). By virtue of (20), the latter can be written as follows:

$$\frac{\partial \eta}{\partial t} = -\frac{1}{2}\mathrm{Im}\left[\frac{1}{x+i+\sqrt{(x+i)^2+t}}\right].$$

After elementary integration of this equation, the following expression for η arises:

$$\eta(x,t) = -\mathrm{Im}\left\{f(x,t) - f(x,0)\right\} + \eta(x,0), \tag{33}$$

where

$$f(x,t) = \sqrt{(x+i)^2+t} - (x+i)\ln\left[x+i+\sqrt{(x+i)^2+t}\right].$$

Again, simple checking shows that both functions $\eta(x,t)$ and

$$\frac{\partial \eta}{\partial x} = \mathrm{Im}\ln\left[\frac{x+i+\sqrt{(x+i)^2+t}}{2(x+i)}\right]$$

are finite up to $t \le t_0$.[Here we omitted the derivative from $\eta(x,0)$.]

Only the second derivative,

$$\eta_{xx} = \mathrm{Im}\left[\frac{1}{\sqrt{(x+i)^2+t}} - \frac{1}{x+i}\right],$$

acquires a singularity at $x = 0$, as $t \to t_0$. Close to the singular point, η_{xx} can be represented in the self-similar form [coinciding with that for $\frac{\partial v}{\partial x}$ (30) except of the multiplier 2]:

$$\eta_{xx} = \frac{1}{\sqrt{\tau}}h\left(\frac{x}{\tau}\right), \tag{34}$$

where

$$h(\xi) = -\sqrt{2}\left[\frac{1+\sqrt{1+4\xi^2}}{1+4\xi^2}\right]^{1/2}.$$

At the critical moment of $\tau = 0$, η_{xx} looks like

$$\eta_{xx} \approx -|x|^{-1/2} \tag{35}$$

that gives after integration the following behavior of η and η_x due to the singularity (35):

$$\eta_x \approx -2\mathrm{sgn}x\,|x|^{1/2} + (\text{regular terms}), \tag{36}$$

$$\eta \approx -\frac{4}{3}|x|^{3/2} + (\text{regular terms}). \tag{37}$$

Thus, the function $\eta(x,t)$ loses its smoothness, as $t \to t_0$, resulting in the appearance of the root singularities on a free surface profile.

If the constant A is an arbitrary complex number, the corresponding branch points, generated by the initial pole, will move under some angle to the real axis. Putting $A = \frac{1}{8}e^{2i\varphi}$ and $a = 1$, one can find that at the moment of $t_0 = \frac{1}{\sin^2\varphi}$ the cut touches the real axis in the point $x_{\rm br} = \cot\varphi$. At the moment of $t = t_0$, quantities v, η, η_x, all remain finite everywhere. At the vicinity of $x_{\rm br}$ and $t \to t_0$ the values $\frac{\partial v}{\partial x}$ and η_{xx} reveal the same asymptotic behavior as considered above.

From this consideration we can conclude that if $v^{(+)}$ has simple poles at the initial moment, then each pole is being transformed into two moving branch points, connected by a cut. The breaking of analyticity occurs when the most "rapid" branch point reaches the real axis. We will show below that close to the touching point the behavior of a solution is the universal one in the framework of the suggested model (15) and (16), and is defined by formulas (31) and (35).

IV. GENERAL SOLUTION

In this section we will show that the type of singularities found before follows from the general initial conditions. Let in $F^{(+)}(x_0)$ (22) be some analytical function in the upper half-plane of complex x_0 with its singularities in the lower half-plane. To find the solution of Eqs. (21) one needs to resolve at first Eq. (23) with respect to x_0 that we rewrite in more convenient form, introducing z and z_0 instead of x and x_0, respectively:

$$z = z_0 + 2F^{(+)}(z_0)t.$$

This is a mapping: $z \to z_0$. In the general case this mapping will be ambiguous if $\frac{\partial z}{\partial z_0} = 0$ in some point, i.e.,

$$1 + 2F^{(+)\prime}(z_0)t = 0. \tag{38}$$

Solution of (38) gives some trajectory on the complex plane z_0: $z_0 = z_0(t)$. The roots of (38) together with (22) define the corresponding movable branch points of the function $v^{(+)}(x,t)$

$$z_{\rm br}(t) = z_0(t) + 2F^{(+)}(z_0(t))t. \tag{39}$$

These points should be connected with a set of cuts, providing for the uniqueness of the function $v^{(+)}(x,t)$. The choice of these cuts has to be made in such a way that $v^{(+)}(x,t)$ would have the initial singularities [coinciding with those for $F^{(+)}(z)$]. As in the pole case, these movable branch points originate from the singularities of the function $F^{(+)}(z_0)$. At the moment when the most rapid branch touches the real axis, the analyticity of $V^{(+)}(x,t)$ breaks, and, respectively, a singularity appears in the solution of system (21). Close to singularity the solution behaves in a similar way as for pole initial conditions. Below we give proof of this fact.

At first define the touching time t_0 from the requirement $z_{\rm br}$ to be real,

$$z_{\rm br} = x_{\rm br}.$$

Assuming $\tau = t_0 - t \ll t_0$, and considering a small vicinity of $z = x_{\rm br}$, expansion of (23) up to the leading orders gives

$$F''t_0(\delta x_0)^2 - 2F'\tau\delta x_0 - 2F_0\tau - x' = 0,$$

where $F'' = F''(z_0(t_0))$, $\delta x_0 = x_0 - z_0(t_0)$, $x' = x - x_{\rm br}$, $F_0 = F^{(+)}(z_0(t_0))$.

From this equation we find

$$x_0 = z_0(t_0) + \frac{F'\tau}{F''t_0} + \sqrt{\left(\frac{F'\tau}{F''t_0}\right)^2 + \frac{2F_0\tau + x'}{F''t_0}}. \tag{40}$$

If $F_0 \neq 0$ the leading term under the square root is the linear one with respect to τ. Therefore with the needed accuracy

$$x_0 = z_0(t_0) + C(x' + 2F_0\tau)^{1/2}, \tag{41}$$

where $C = \{F^{(+)\prime\prime}[z_0(t_0)]\}^{-1/2}$.

Such a general form for x_0 provides in this case the same self-similar ($x' \sim \tau$) singular dependences for $\frac{\partial v}{\partial x}$ and η_{xx}. These follow after the substitution of (41) into (22) and the forthcoming integration of Eq. (18). With the same accuracy as in (41), the tangent velocity can be presented in the form

$$v = 2{\rm Re}\left(F_0 - \frac{1}{t_0}C(x' + 2F_0\tau)^{1/2}\right). \tag{42}$$

The answer for $\partial v/\partial x$, following from this expression, coincides with that for simple poles (up to some constant). To find η_{xx} one should integrate Eq. (18) for $\eta^{(+)}$:

$$\eta^{(+)} = -i\int_0^t F^{(+)}(x_0)dt,$$

where the dependence $x_0(x,t)$ is defined by means of (23). If one translates from the integration over t to the integration over x_0 and then integrates once by parts, the expression for $\eta^{(+)}$ can be written in the form

$$\eta^{(+)} = -i\left(tF^{(+)}(x_0) - \int_x^{x_0(x,t)} \frac{(x - x_0)}{2F^{(+)}(x_0)} \times F'^{(+)}(x_0)dx_0\right).$$

Now differentiate $\eta^{(+)}$ with respect to x. Then the integration can be performed completely. As a result we have the explicit expression for η_x:

$$\eta_x = {\rm Im}\log\frac{F^{(+)}(x)}{F^{(+)}(x_0)}, \tag{43}$$

where $x_0 = x_0(x,t)$ is determined with the help of relation (23). This formula together with (42) lead to the previous answer which we obtained for simple poles: η_{xx} and v_x become infinite while approaching the singularity, $x' \sim \tau$. It is the general type of singularities for systems (17) and (18).

V. WEDGE-TYPE SINGULARITIES

Let us show that the systems (17) and (18) have a special solution that describes another type of singularity. This solution arises if $F_0 = 0$. For this particular case formula (40) transforms into

$$x_0 = z_0(t_0) + \frac{F'\tau}{F''t_0} + \sqrt{\left(\frac{F'\tau}{F''t_0}\right)^2 + \frac{x'}{F''t_0}}$$

and, as a sequence, v can be written approximately in the form

$$v \approx [x_0 - z_0(t_0)]F'. \quad (44)$$

These dependences give a new kind of self-similar behavior $x \sim \tau^2$ that provides the surface singularity of the wedge type. Indeed, let us substitute (43) into (44) and consider the asymptotics of η_x for $x'/\tau^2 \to \infty$. As a result we get

$$\eta_x \to -\frac{\pi}{4}\mathrm{sgn}(x'),$$

which corresponds to the wedge surface profile with the angle $\alpha = 2\arctan\frac{4}{\pi} \approx 103,7°$. This angle is far from π and our assumption about small surface angles breaks, of course, close to the singularity. Nevertheless, the solution obtained above, which represents the intermediate asymptotics, is of considerable interest because, first of all, the angle α is close to that calculated by Stokes for the critical stationary gravity surface wave for deep water and, secondly, the self-similarity $x \sim \tau^2$ remains for the complete system of Eqs. (7) and (8). It is interesting to note that $F_0 = 0$ can be obtained from the initial conditions with two poles:

$$F^{(+)}(z) = i\mu\left[\frac{a}{z+ia} - \frac{a^*}{z+ia^*}\right],$$

where $\mathrm{Re}a > 0$, $\mathrm{Im}\mu = 0$.

The dynamics of the branch points generated by these two poles is also interesting: at the first moment of time poles produce two pairs of branch points, two of which move towards imaginary axis; then they collide; after collision points move along imaginary axis in opposite directions; the touching of the real axis by one of these poles produces the singularity appearance.

VI. FLOATING SINGULARITIES

The new singularities are connected with a possibility of exact integration of Eq. (14) taking into account the second term in its rhs. For this reason let us represent $\eta(x,t)$ as a sum of two functions $\eta^{(+)}(x,t)$ and $\eta^{(-)}(x,t)$. Then from (14) follows an equation for $\eta^{(+)}$,

$$\frac{\partial\eta^{(+)}}{\partial t} + 2\hat{P}^{(+)}(v^{(-)}\eta^{(+)})_x = -i\Psi_x^{(+)}. \quad (45)$$

Furthermore, it is convenient to introduce instead of $\eta^{(+)}$ a new function $\xi^{(+)}$ by means of $\eta^{(+)} = \frac{\partial\xi^{(+)}}{\partial x}$ and to integrate (45) once

$$\hat{P}^{(+)}\left(\frac{\partial\xi}{\partial t} + 2v^{(-)}\frac{\partial\xi}{\partial x}\right) = -i\Psi^{(+)}. \quad (46)$$

Here ξ is a function, for which $\hat{P}^{(+)}\xi = \xi^{(+)}$. Omitting then in both sides of (46) the operator $\hat{P}^{(+)}$, we arrive at the equation for ξ

$$\frac{\partial\xi}{\partial t} + 2v^{(-)}\frac{\partial\xi}{\partial x} = -i\Psi^{(+)} + \Phi^{(-)}, \quad (47)$$

where $\Phi^{(-)}$ is some unknown function, for which $\hat{P}^{(+)}\Phi^{(-)} = 0$. Equation (47) [compare with (21)] can be integrated with the help of the characteristics, defined by (23):

$$x = x_0 + 2F^{(-)}(x_0)t. \quad (48)$$

On the characteristic x_0

$$\xi = \xi_1 + \xi_2,$$

where

$$\xi_1 = -i\int_0^t \psi^{(+)}(x(x_0,t'),t')dt' + \int_0^t \Phi^{(-)}(x(x_0,t'),t')dt'$$

is the solution of the homogeneous equation and $\xi_2 = f(x_0)$ is simply the initial shape of ξ.

Thus, the problem is separated into two parts and, consequently, there will appear two possibilities. The first one is defined by the analytical properties of the tangent velocity only. As will be seen below, the singularities that arise are almost the same as found in the previous section. The second possibility is connected with the contribution of ξ_2 that gives rise to the interference of the tangent velocity effect and intrinsic peculiarities of the initial elevation $\eta_0(x)$.

We start to analyze the first contribution to ξ. It is easy to understand that the integration of the function $\Phi^{(-)}$ along characteristics (48) with forthcoming application of the operator $\hat{P}^{(+)}$ equals zero. It is enough, therefore, to integrate only $\Psi^{(+)}$ in (47). In fact, the situation is even more simple, because we are interested in the solution behavior only close to the moment of t_0. This implies that instead of (48) one may use its expansion

$$x' = x_0' - 2\bar{F}_0\tau - \frac{2C\tau}{t_0}(x_0')^{1/2}, \quad (49)$$

where x_0' is the coordinate x' at $\tau = 0$. Formula (49) shows that close to the singularity some additional motion arises, as compared to case (18). Nevertheless, the character of singularity remains the same:

$$\eta_{1xx} = -\frac{1}{8t_0\mathrm{Im}F_0}\,\mathrm{Re}[C(x' + 2F_0\tau)^{-1/2}],$$

yielding at $t = t_0$,

$$\eta_{xx} \sim |x'|^{-1/2}.$$

It is very important that the singularities obtained belong to the weak ones [see (36) and (37)], which do not destroy our basic assumption about small values of angles, $|\nabla\eta| \ll 1$. Note also that the self-similar asymptotics of the form (34) is admitted by the complete set of Eqs. (7) and (8).

We now proceed to the homogeneous part of the solution $\xi_2 = f(x_0)$. One can show that the elevation $\eta^{(+)}$ is defined as

$$\eta_2^{(+)}(x,t) = \left[\frac{\partial x_0}{\partial x}\frac{df}{dx_0}\right]^{(+)}.$$

Since at the initial moment of $t = 0$,

$$x = x_0, \quad \frac{\partial x_0}{\partial x} = 1$$

the exact form of $\eta_2^{(+)}$ may be written as follows:

$$\eta_2^{(+)}(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dx'}{x' - x - i0} \quad \frac{\partial x_0(x',t)}{\partial x'} \quad \eta_0^{(+)}(x_0).$$

Passing to x_0 as a new variable of integration, this integral reduces to the form

$$\eta_2^{(+)}(x,t) = \frac{1}{2\pi i}\int_C \frac{dx_0}{x'(x_0,t) - x - i0}\eta_0^{(+)}(x_0) \qquad (50)$$

with x' and contour C, both defined from (48). The contour C initially coincides with the real axis during the time it is being deformed so that it is going partly through the lower half-plane. The motion of contour C towards singular points of $\eta^{(+)}(x_0)$ will define obviously the behavior and the singularity formation of function $\eta(x,t)$ for real x. To clarify this situation let us assume that $\eta^{(+)}(x_0)$ has one pole in the lower half-plane,

$$\eta^{(+)}(x_0) = \frac{iB}{x_0 - b},$$

where B is real, and $\text{Im}b < 0$. Then integral (50) is found explicitly

$$\eta_2^{(+)}(x,t) = \frac{iB}{x - x'(b,t)} = \frac{iB}{x - b - 2F^{(-)}(b)t}.$$

From this expression we see that the pole of $\eta^{(+)}$ is movable with the "velocity" $2F^{(-)}(b)$, being some regular function. Therefore if $\text{Im}F^{(-)}(b) > 0$ then there exists such moment of time t_c when $\eta_2(x,t)$ will be infinite. Evidently

$$t_c = -\frac{b''}{2F''(b)}.$$

Here $b'' = \text{Im } b, F''(b) = \text{Im } F(b)$.

Close to this time $\eta(x,t)$ has the Lorenz form,

$$\eta(x,t) = -\frac{B(b'' + 2F''t)}{[x - b' - 2F'(b)t]^2 + (b'' + 2F''t)^2},$$

which transforms at $t = t_c$ into the δ function:

$$\eta(x,t) = B\pi\delta\left(x - b' + b''\frac{F'}{F''}\right).$$

Thus, the proper singularities of the analytical function $\eta^{(+)}$, not generated by the velocity field and existing initially, remain during the time and occur to be movable. (It easy to check this statement for an arbitrary case and not only for poles.) It gives new types of singularities of the free surface, generally speaking, of the arbitrary kind appearing due to the proper analytical properties of the initial profile of the elevation. What kinds of singularities will appear first depends on the initial conditions. If, for instance, the initial elevation is equal to zero then we get the first kind of singularities of the root character. One should pay attention to the fact that for the second kind of singularities our assumption about small surface angles breaks. Close to the time $t = t_c$ we should use instead of reduced Eqs. (14) and (15) the complete system (7) and (8).

VII. CONCLUDING REMARKS

In conclusion, we would like to pay attention to the symmetry properties of Eq. (15) or its analog (21). If one introduces instead of $v^{(+)}$ two new functions, namely, its real and imaginary parts, $v^{(+)} = u/2 + in$, then Eq. (21) (for real x) transforms into the system (compare [9], [10]):

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \qquad (51)$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(2n^2) = 0, \qquad (52)$$

which describes a gas with negative pressure. In such a gas dynamic representation the quantity n playing the role of the "gas" density will increase due to the negative pressure. At first glance, such gas has to be compressed, and a solution of Eqs. (51) and (52) appears to form a singularity of the collapse type with density n turning into infinity in a finite time. Such speculation, however, seems to be irrelevant, because the analyticity violation takes place earlier than formation of collapse. In this sense the requirement of analyticity proves to be more essential for the dynamics of systems (17) and (18). It is just the analyticity breaking that leads to the singularity, corresponding to a usual wave breaking in gas dynamics.

It is worth noting that systems (51) and (52) may be obtained as a semiclassical limit of the nonlinear Schrödinger equation (NLSE)

$$i\Psi_t + \frac{1}{2}\Psi_{xx} + |\Psi|^4\,\Psi = 0, \qquad (53)$$

where the wave function Ψ is connected with "density" n and "velocity" u by the following formulas:

$$\Psi = \sqrt{n}\,e^{i\Phi}, \quad u = \Phi_x.$$

It should be underlined, however, that the analogy with the semiclassical NLSE is not complete. Unlike Eq. (53), where the quantity $|\Psi|^2 = n$ is always positive, the "density" n in Eqs. (51) and (52) may have either sign. Nevertheless, the found correspondence between the two systems reveal many novel features of (51) and (52). In particular, one can write the virial theorem [13];

$$\frac{\partial^2}{\partial t^2}\int x^2 n\, dx = 4H, \qquad (54)$$

where

$$H = \int\left[\frac{nu^2}{2} - \frac{n^3}{3}\right]dx$$

is the Hamiltonian for systems (51) and (52). Since the NLSE of the form (53) relates to the so-called critical one [the latter means that the collapse takes place only starting from the power $|\Psi|^4$ in (53)], systems (51) and (52) inherit from (53) two additional symmetries of the Noether type [14] and [15]. These are scaling transformations

$$n(x,t) = \frac{1}{\lambda}\tilde{n}(\lambda x,\ \lambda^2 t), \tag{55}$$

$$u(x,t) = \frac{1}{\lambda}\tilde{u}(\lambda x,\ \lambda^2 t), \tag{56}$$

and "lens" transformation,

$$n(x,t) = \frac{t_0}{t_0 - t}\,\tilde{n}(x',t'), \tag{57}$$

$$u(x,y) = -\frac{x}{2(t_0-t)} + \frac{t_0}{t_0-t}\tilde{u}(x',t'), \tag{58}$$

where $x' = xt_0(t_0-t)^{-1}$, $t' = tt_0(t_0-t)^{-1}$, and t_0 and λ are arbitrary parameters.

Transformations in the form (55)—(58) mean that $\tilde{n}$ and $\tilde{u}$ obey the same equations as n and u do. Besides, these transformations remain active to be invariant and, therefore, generate new integrals of motion. They appear just as one integrates twice Eq. (54) (compare with [15]). It should be noted that such types of symmetries have been known in usual gas dynamics, in fact for hyperbolic cases. Use of them has been made in both implicit and explicit forms in papers [11], [12]. It is unlikely that systems (51) and (52) belong to the elliptic type that manifests in its negative pressure.

In this paper we did not touch such a question as the stability problem of the collapsing regimes. The first regime of the root character according to the analysis performed in Sec. IV will be obviously stable in the framework of truncated systems (17) and (18). For the complete system, however, this is the question as well as for two other regimes. It should be emphasized again that from the very beginning we assumed the angle of the surface ($|\nabla\eta|$) to be small, and therefore, we cannot pretend to understand the full description of all types of possible singularities, as described by the complete system of Eqs. (7) and (8). However, the solutions corresponding to the weak singularity regime turn out to be consistent with the applicability condition of the truncated equations (15) and (16).

In our opinion there exist two possibilities of what role the root singularities will play in the general dynamics—either the singularities serve as an origin of more powerful singularities observed in numerical experiments or represent new type of singularities.

One should also note that the self-similar asymptotics for the wedge type of singularities are allowed by the exact system of equations. We believe therefore that just this type of singularity was observed in numerical experiments [7] (see also [6]).

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